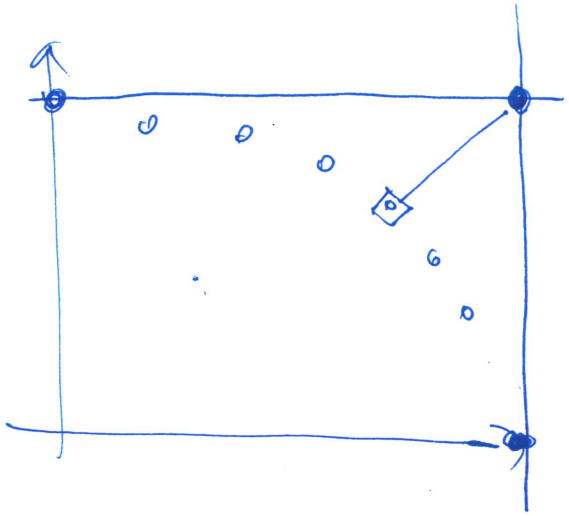


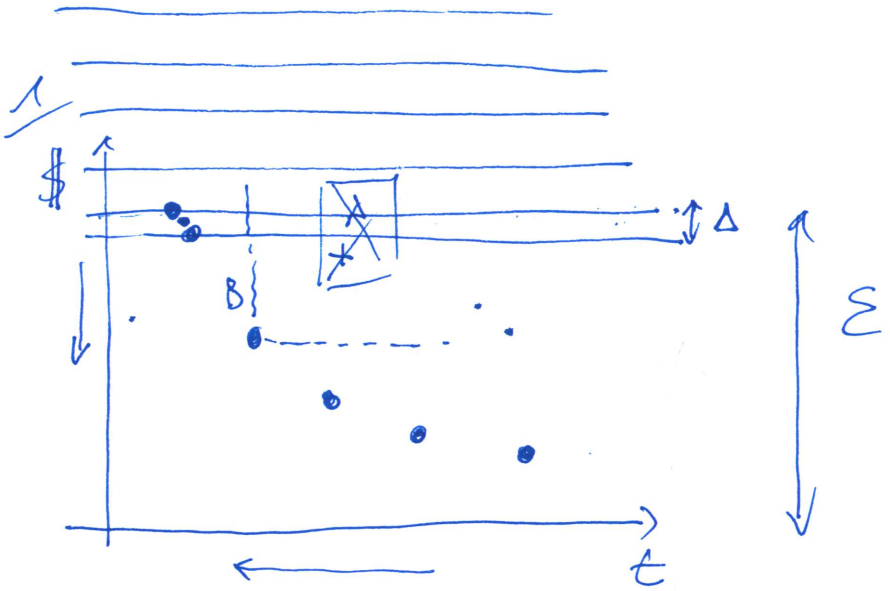
D.	10	9	8	7	6	5
S1	3	2	1	0	0	
S2	3	3	2	0	0	

S1 < S2

+ good perspective

– machine learning – these it in intro
several MC courses

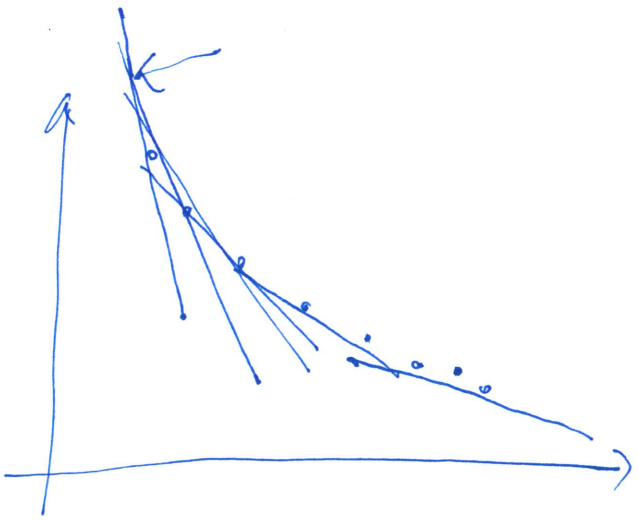
CP + LS are in that book

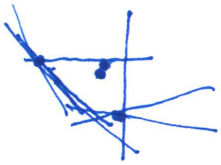


$[x_i^A]$ $\quad i = 1 \dots m$

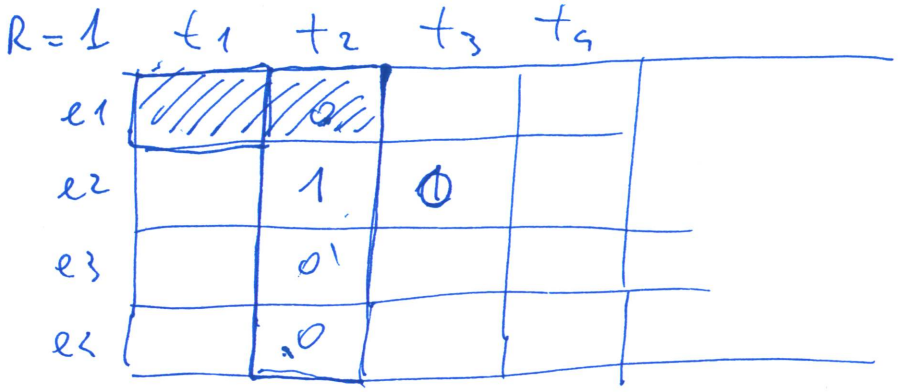
$\vec{x}^A \prec \vec{x}^B$ if $x_i^A < x_i^B \quad \forall\, i = 1 \dots m$

$\vec{x}_A \equiv \vec{x}_B$ if $\vec{x}_i^A \not< \vec{x}_i^B$





$\alpha f_1 + (1-\alpha) f_2$



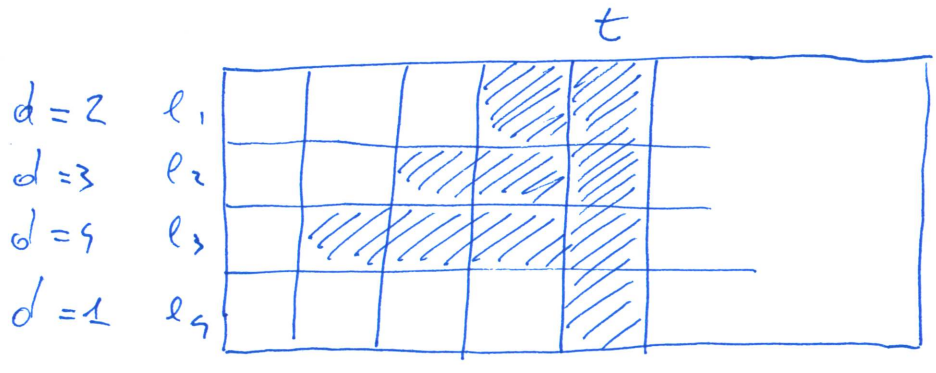
$$x_{e_1,t_2,1} + x_{e_2,t_2,1} + x_{e_3,t_2,1} \leq 1$$

$$x_{e_1,t_2,1} + x_{e_1,t_2-1,1} +$$

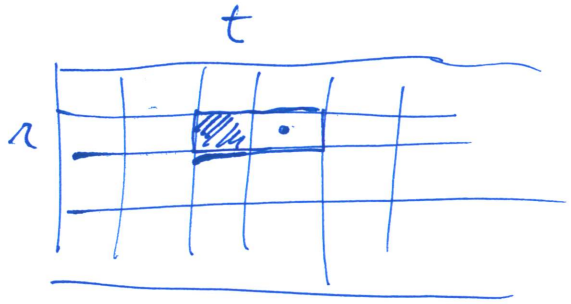
$$+ x_{e_2,t_2,1} + x_{e_2,t_2-1,1} + \cancel{x_{e_2,t_2-2,1}}$$

$$\sum_{l=\max(0,\, t_2 - d(e_2)+1)}^{t_2} x_{e_2,l,1}$$

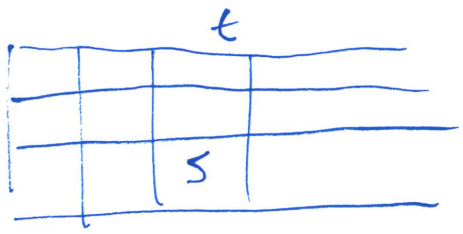
$$\sum_{e \in E} \sum_{l=\max(0,\, t-d(l)+1)}^{t} x_{e,l,r} \leq 1 \qquad \forall t \in T \quad \forall r \in R$$



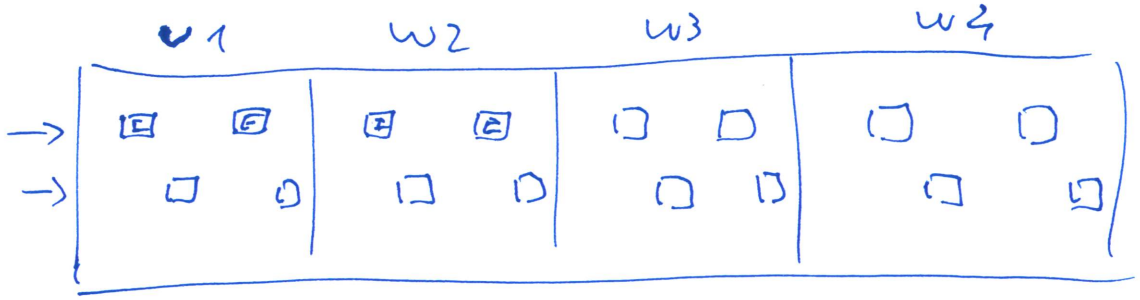
X_{itr}

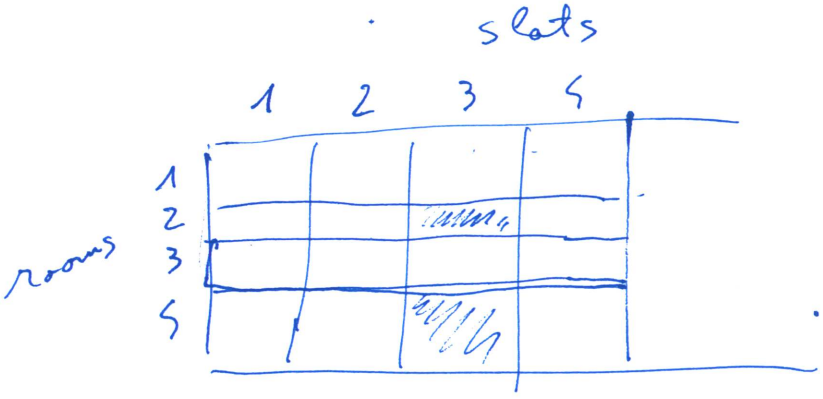


$X_{tr} \in E$

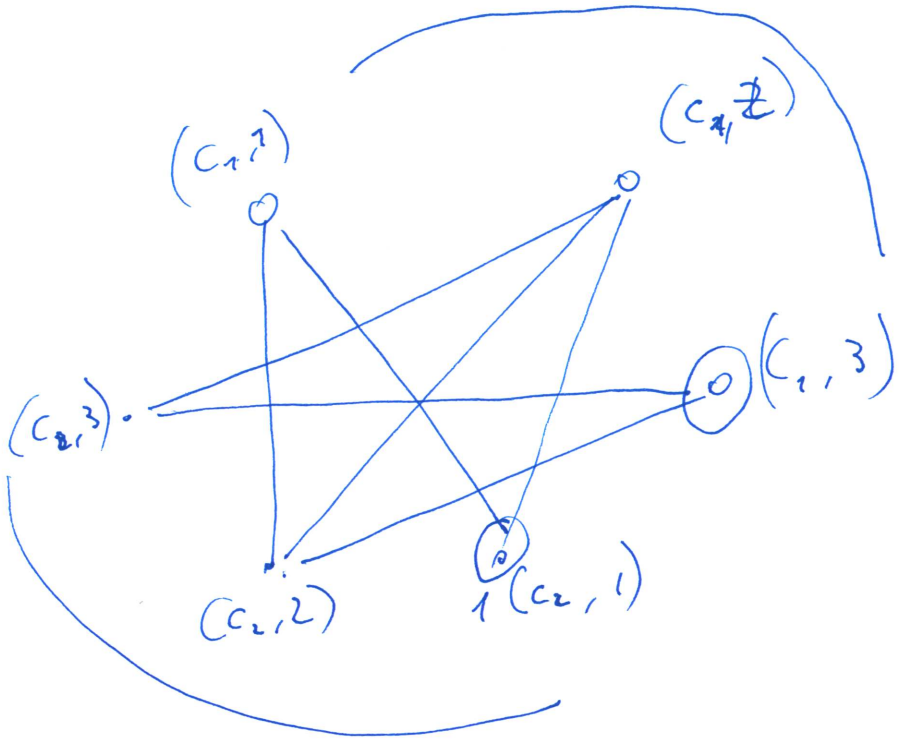


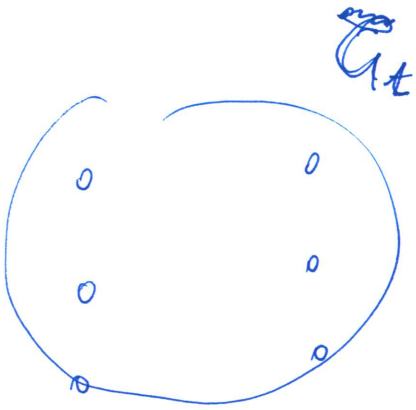
X_p





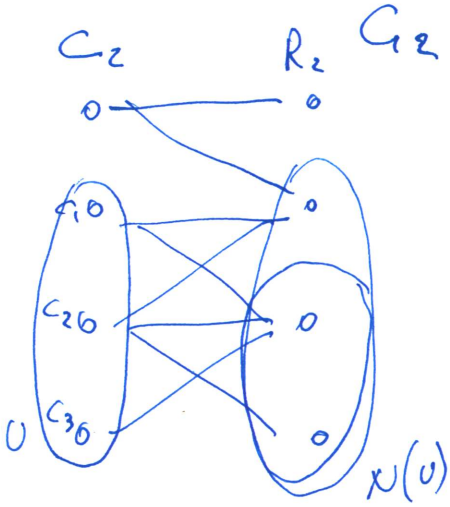
$X_{13t} + X_{23t} + X_{33t} + X_{43t} \leq 1$

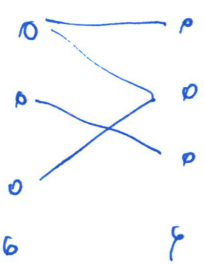




$$\sum_{i=\{c_1,c_2,c_3\}} x_{i2} \leq 2$$

U — $N(U)$

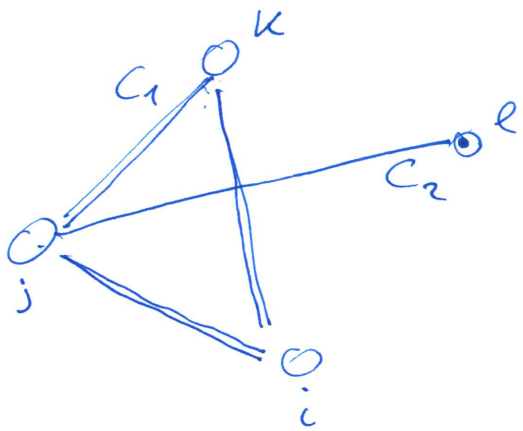




$$G = \bigcup_i G_i$$

$\{i,j,k\}$ $\{j,l\}$

$\mathcal{C} = \{C_1, C_2\}$

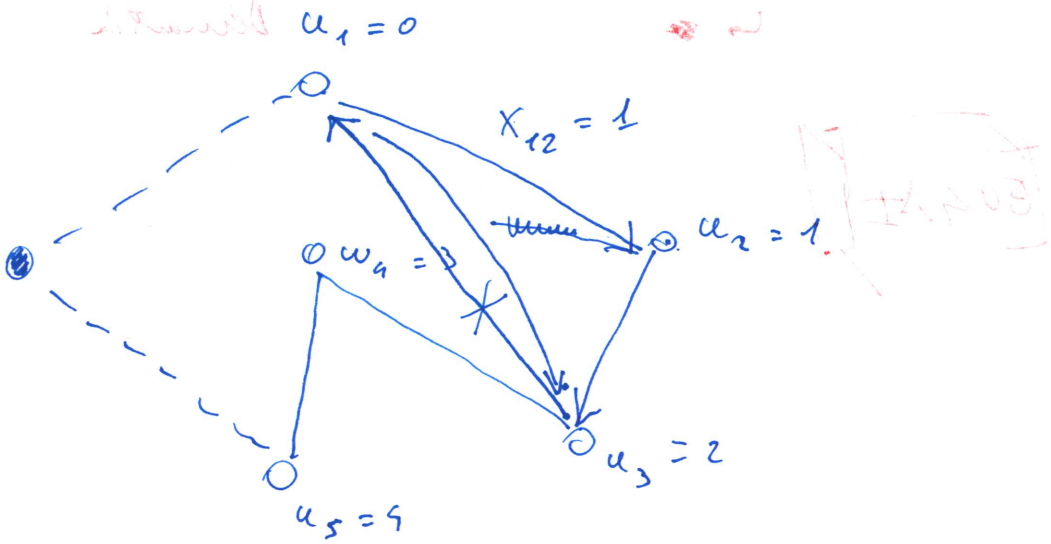


$$\sum_{i \in C_q} x_{it} \leq 1 \quad \forall\, C_q \in \mathcal{C}$$

$$x_{it} + x_{jt} \leq 1 \quad \forall\, ij \in E \quad \forall\, t = 1 \ldots p$$

$$x_{it} + x_{jt} + x_{kt} \leq 1 \quad : C_1$$

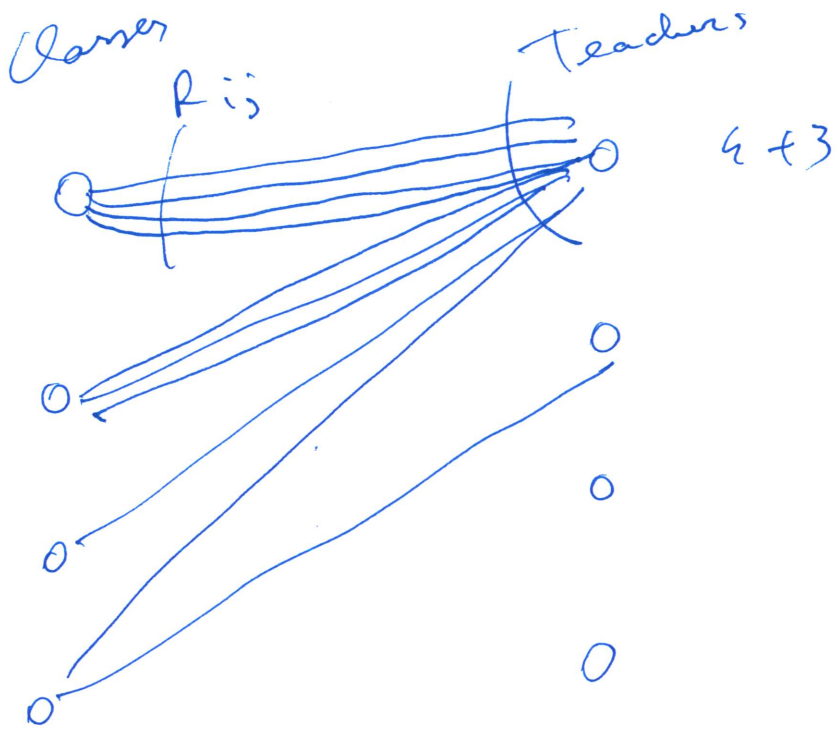
$$x_{jt} + x_{lt} \leq 1 \quad : C_2$$



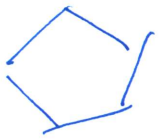
$$u_{i+1} \gtrless u_j + M\,(1 - x_{ij})$$

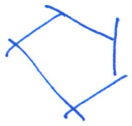
$$\begin{cases} u_i + 1 \gtrless u_j & \text{if } x_{ij} = 1 \\ u_i + 1 \leq u_j + M & \text{if } x_{ij} = 0 \end{cases} \Longleftarrow$$

$$1 \leq u_i \leq M - 1$$


Classes
Rij
Teachers
4+3

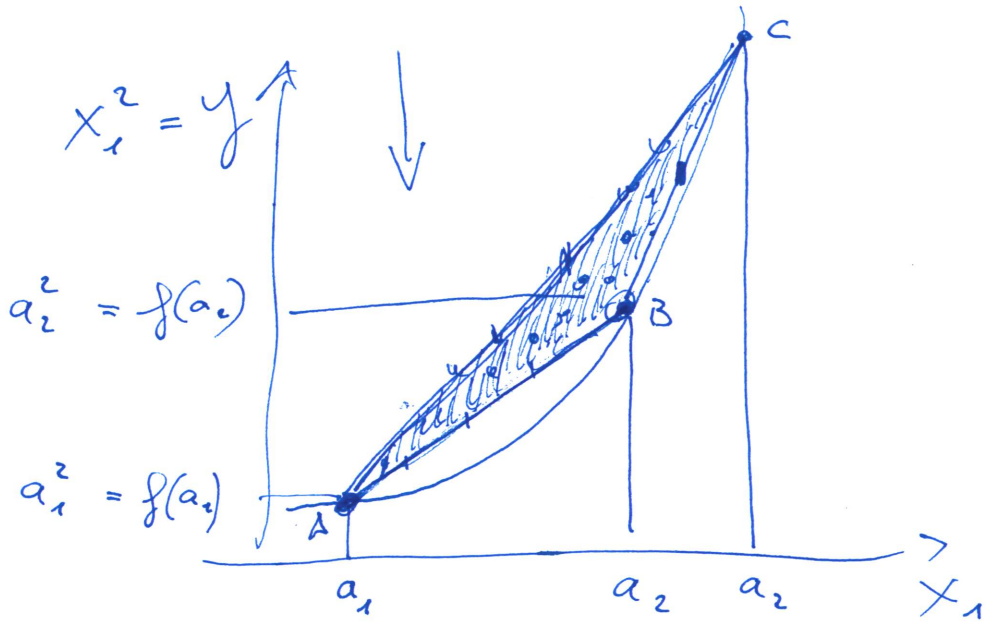
$A_1 x \leq b_1$ $\vee$ $A_2 x \leq b_2$





$A_1 x \leq b_1 + My$

$A_2 x \leq b_2 + (1-y)M$ $\quad y \in \{0,1\}$

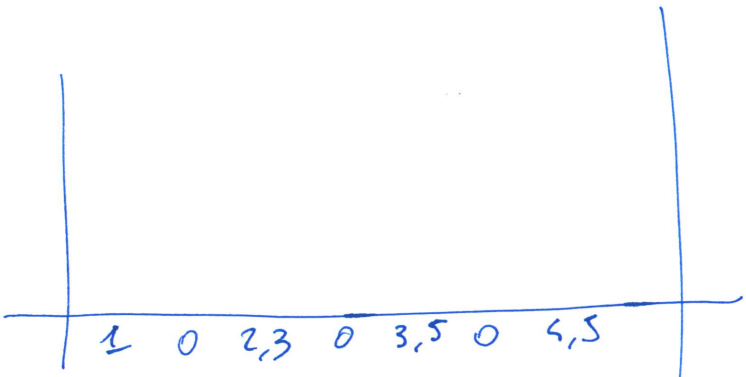


$$\vec{v}_1 = \begin{bmatrix} a_1 \\ f(a_1) \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} a_2 \\ f(a_2) \end{bmatrix}$$

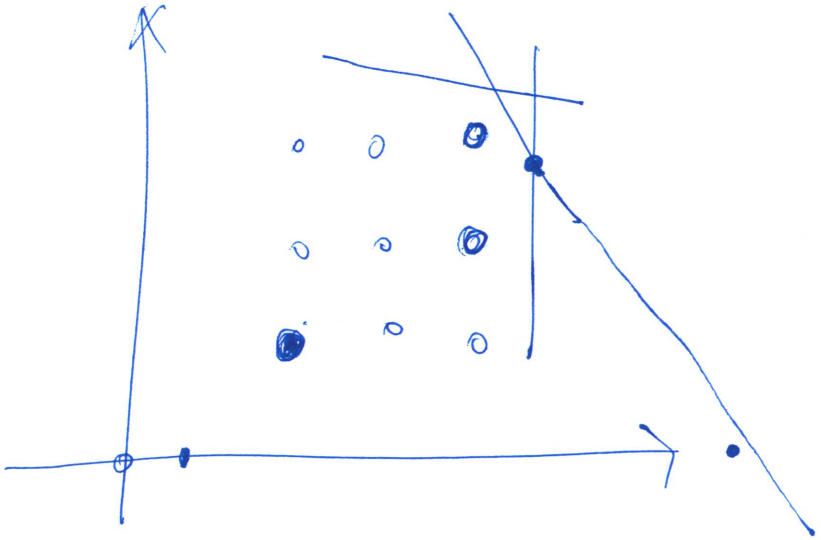
$$\begin{cases} \vec{z} = \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 \\ \lambda_1 + \lambda_2 = 1 \\ \lambda_1, \lambda_2 \geq 0 \end{cases} \qquad z = \begin{bmatrix} x_* \\ y_* \end{bmatrix}$$

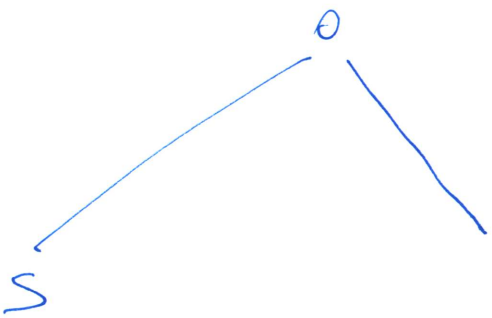
$$\left[\begin{cases} x = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 \\ y = \lambda_1 f(a_1) + \lambda_2 f(a_2) + \lambda_3 f(a_3) \\ \sum_{i=1}^{3} \lambda_i = 1 \\ \lambda_i \geq 0 \quad \forall i = 1,2,3 \end{cases} \right] \Leftarrow$$

$\Rightarrow$ at most two consecutive λs are $\neq 0$ $\Leftarrow$

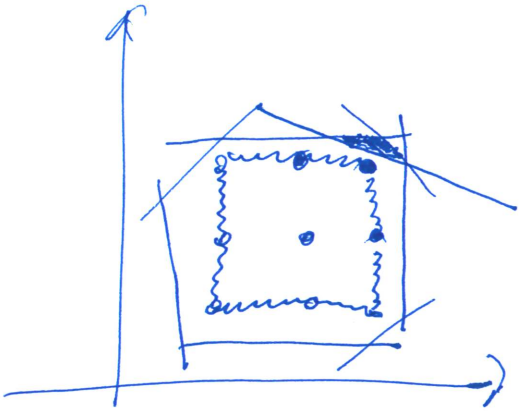


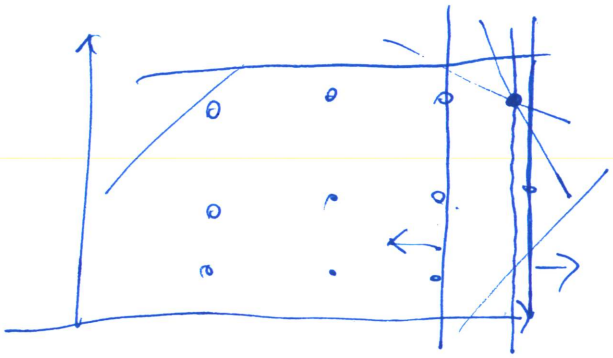
0 0 0 0 0 1 2,3 | 3,5 4,5

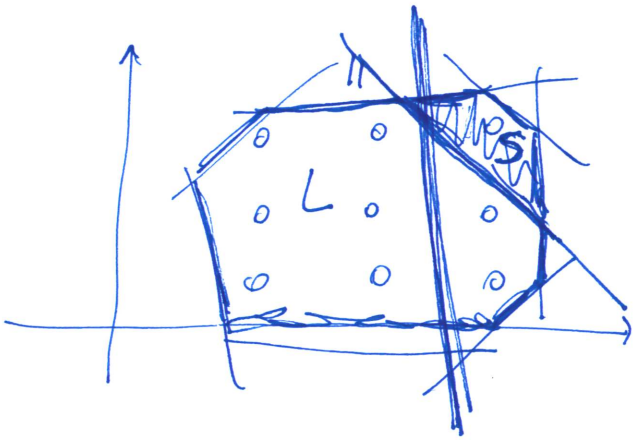




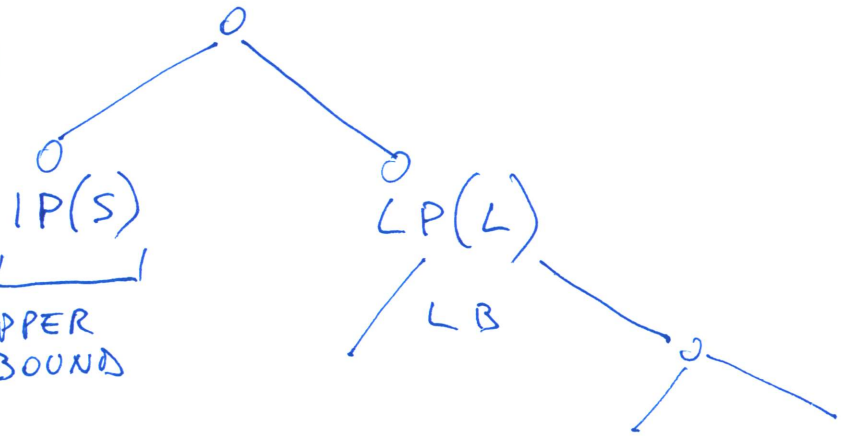
$X_i \in L = 0$ $\quad \sum_{X_i \in L} X_i \geq 1$







MIN PROBLEM



UB = UPPER BOUND

if LB ≥ UB ⇒ OPT is found

$$\sum_{e \in E(S)} x^*_e \geq |S \setminus \{u\}| = |S| - 1$$

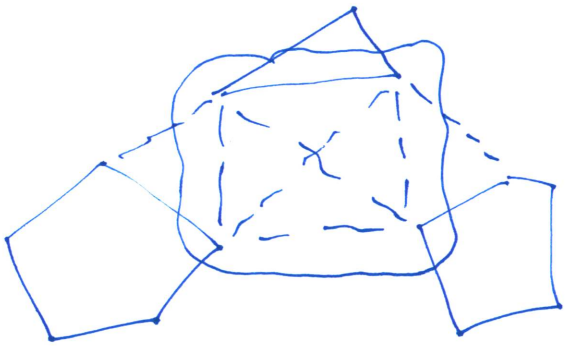
$$z_i \in \{0,1\} \quad \forall i \in V$$

$$\xi^u = \max \left\{ \sum_{ij \in E \setminus \{0,i\}} x^*_{ij} z_i z_j - \sum_{i \in V \setminus \{0\}} z_i \right\}$$

$$S = V \setminus \{0\}$$

$$\boxed{\xi^u \leq 0}$$

$$\xi^u > 0$$



$$\max \sum x^*_{ij} y_{ij} - \sum z_i$$

$$y_{ij} \leq z_i$$

$$y_{ij} \leq z_j$$

$$z_i + z_j \geq y_{ij} + 1$$

$$z_i, z_j, y_{ij} \in \mathbb{R}$$

z_i	z_j	y_{ij}
0	0	0
1	0	0
0	1	0
1	1	1

$$\min(z_i, z_j)$$

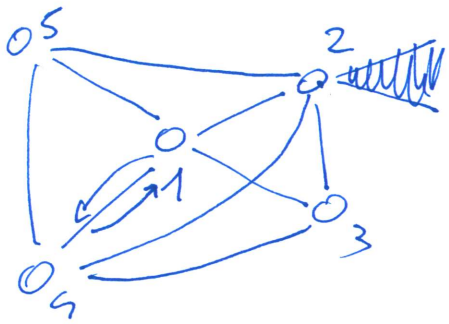
$$\min(z_i, z_j) \geq z_i + z_j - 1$$

0	$\geq$	0
1	$\geq$	1
0	$\geq$	-1

$$\min c^T x$$

$$Ax = \vec{2}$$

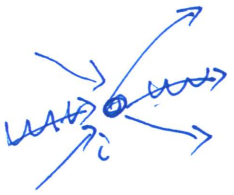
$$x \in \mathbb{R}^+$$



	(12)	(13)	(14)	(15)	(23)	(24)	(25)	
1:	1	1	1	1	0	0	0	= 2
2:	1	0	0	0	1	1	1	= 2
3:	0	1	0	0	1	0	0	= 2
4:	0	0	1	0	0	1	0	= 2
5:	0	0	0	1	0	0	1	= 2

m.pprint()

$$\sum_{ij \in E} x_{ij} = 2$$



$$\sum_{ij \in \delta^+(i)} x_{ij} = 1 \qquad \forall i \qquad \delta^+(i) \subseteq A$$

$$\delta^+(i) = \{ ij \in A \mid j \in V \}$$

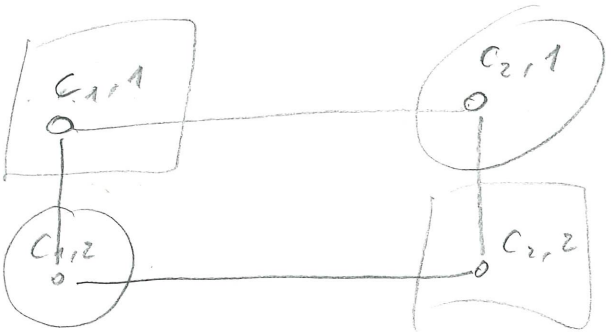
$$\sum_{ij \in \delta^-(j)} x_{ij} = 1$$

$$\boxed{\sum_{ij \in \delta^+(i)} x_{ij} - \sum_{ij \in \delta^-(ij)} x_{ij} = 0}$$

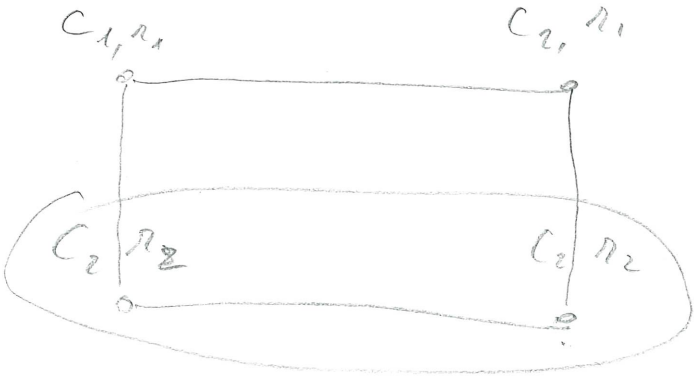
V_1 V_2

1		1
0		0
0		0
0		0
-1		0
0		-1
0		0

dans, time.



$$\sum_{n} x_{c_{1,1},n} + \sum x_{c_{2,1},n} \leq 1$$

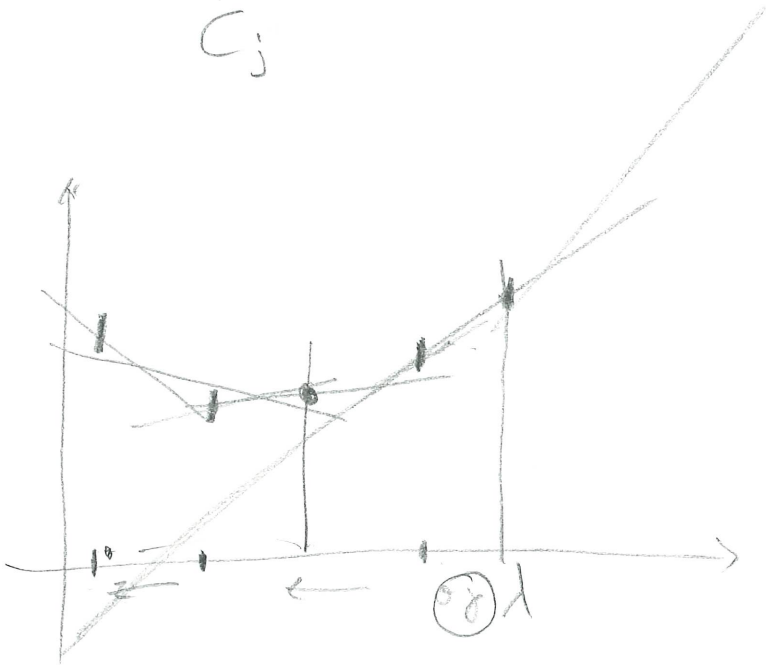


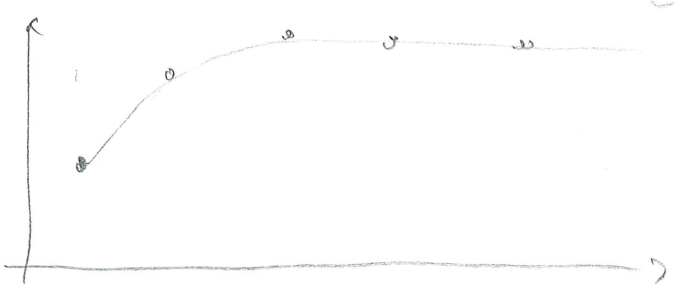
$$x_{c_1 n_1, t} + x_{c_2 n_1, t} \leq 1 \qquad \forall t \in T$$

$$\min \quad \sum c_j x_j \; - \lambda \left(\sum a_{ij} x_j - 1 \right)$$

$$\left(\sum_j a_{ij} x_j - 1 \right) \geq 0$$

$$\min \quad \sum_j \underbrace{\left(c_j - \lambda a_{ij} \right)}_{c_j} x_j + \lambda$$

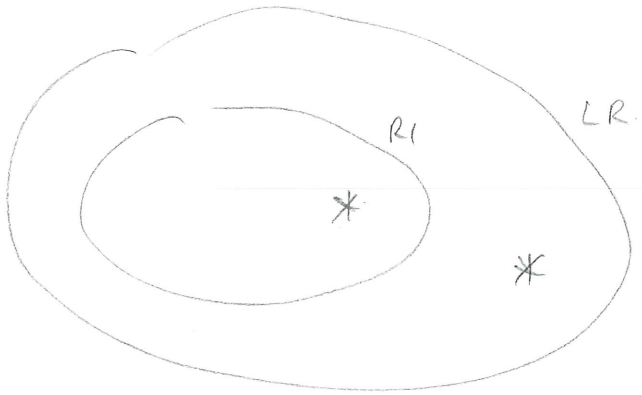




$$z_{LR} = \min \quad cx + \lambda(Ax - b)$$
$$Bx \leq d$$
$$x \in \{0,1\}$$

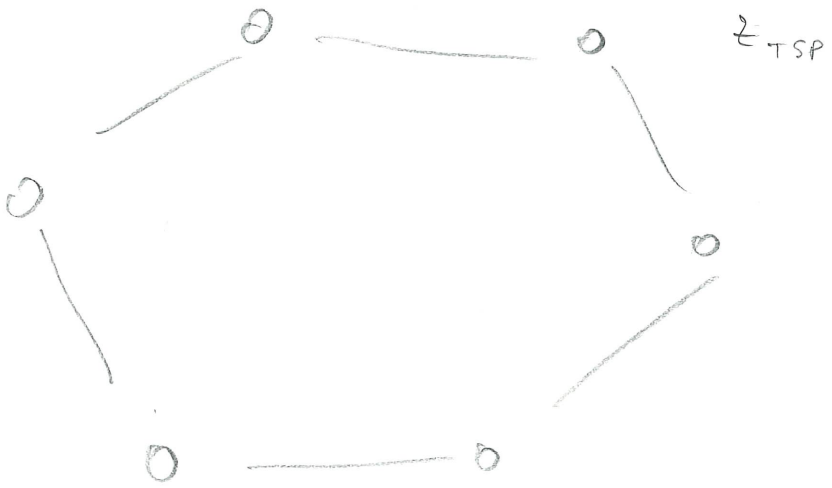
$$cx = cx + \lambda \underbrace{(Ax - b)}_{\leq 0}$$

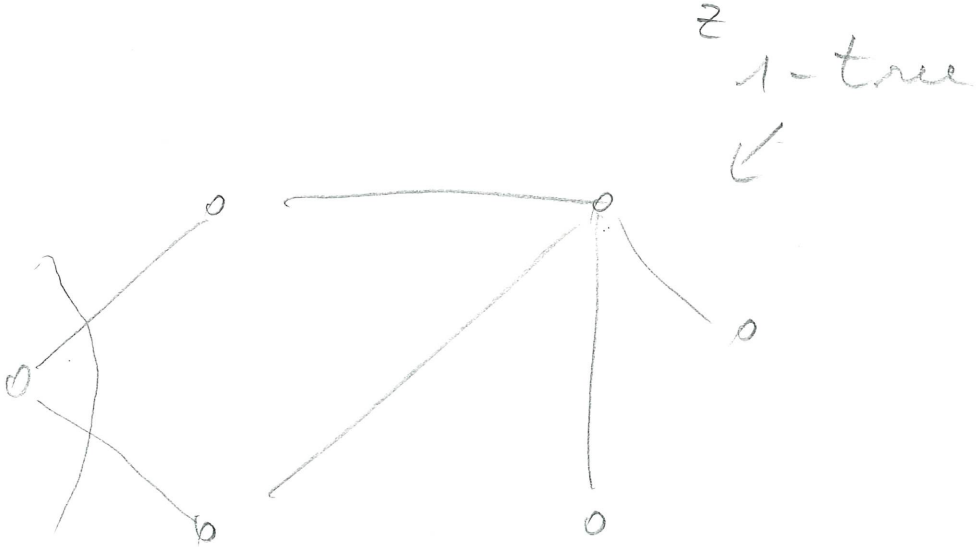
$$\lambda(Ax - b) = 0$$



$Ax = b$

$$cx = cx + \underbrace{\lambda(Ax - b)}$$





$$z_{1\text{-}tree} \leq z_{TSP}$$

$z =$

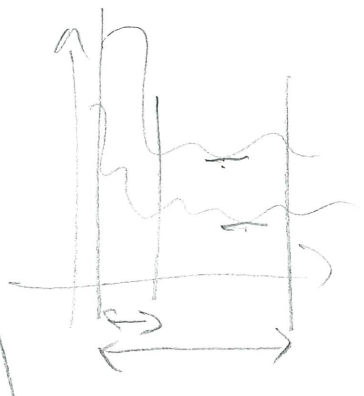
$z_{R1} = \min\ 2x_1 + 3x_2 + 4x_3 + 5x_4 +$

$-\lambda_1 (x_1 + x_3 - 1) +$

$-\lambda_2 (x_1 + x_4 - 1) +$

$-\lambda_3 (x_2 + x_3 + x_4 - 1)$

$\lambda \geq 0$



$g(x) \leq f(x)$

$z_{R1} \leq z$

$$\begin{pmatrix} x_1 + x_3 \geq 1 \\ x_1 + x_4 \geq 1 \\ x_2 + x_3 + x_4 \geq 1 \end{pmatrix}$$

$x_i \in \{0, 1\}$

$z_{LR} = \min\ 2x_1 + 3x_2 + 4x_3 + 5x_4$

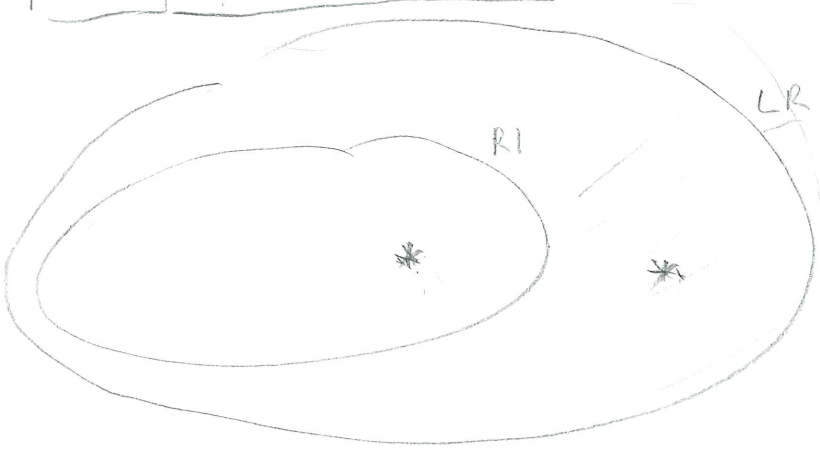
$-\lambda_1 (x_1 + x_3 - 1)$

$-\lambda_2 (x_1 + x_4 - 1)$

$-\lambda_3 (x_2 + x_3 + x_4 - 1)$

$x_i \in \{0, 1\}$

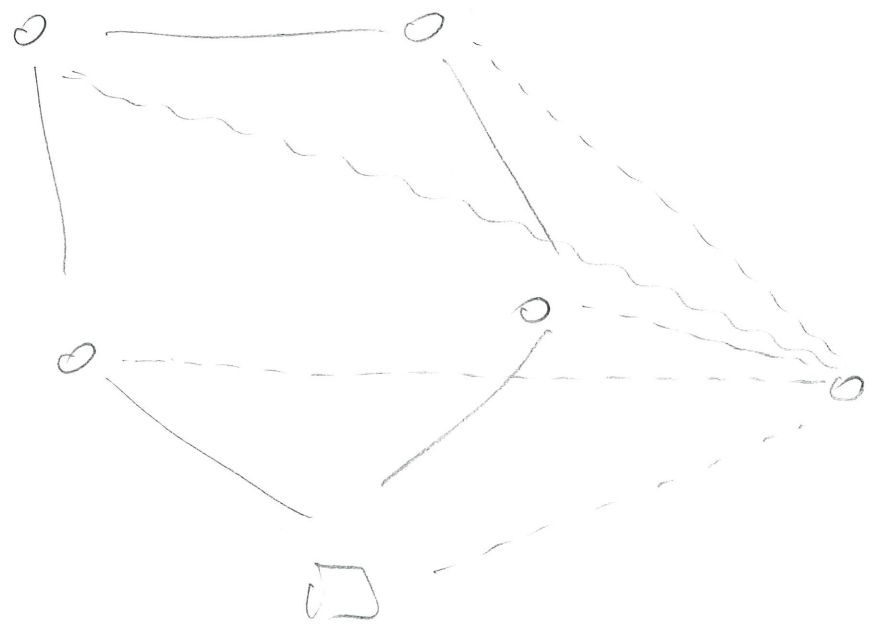
$$z_{LR} \leq z_{R1} \leq z$$

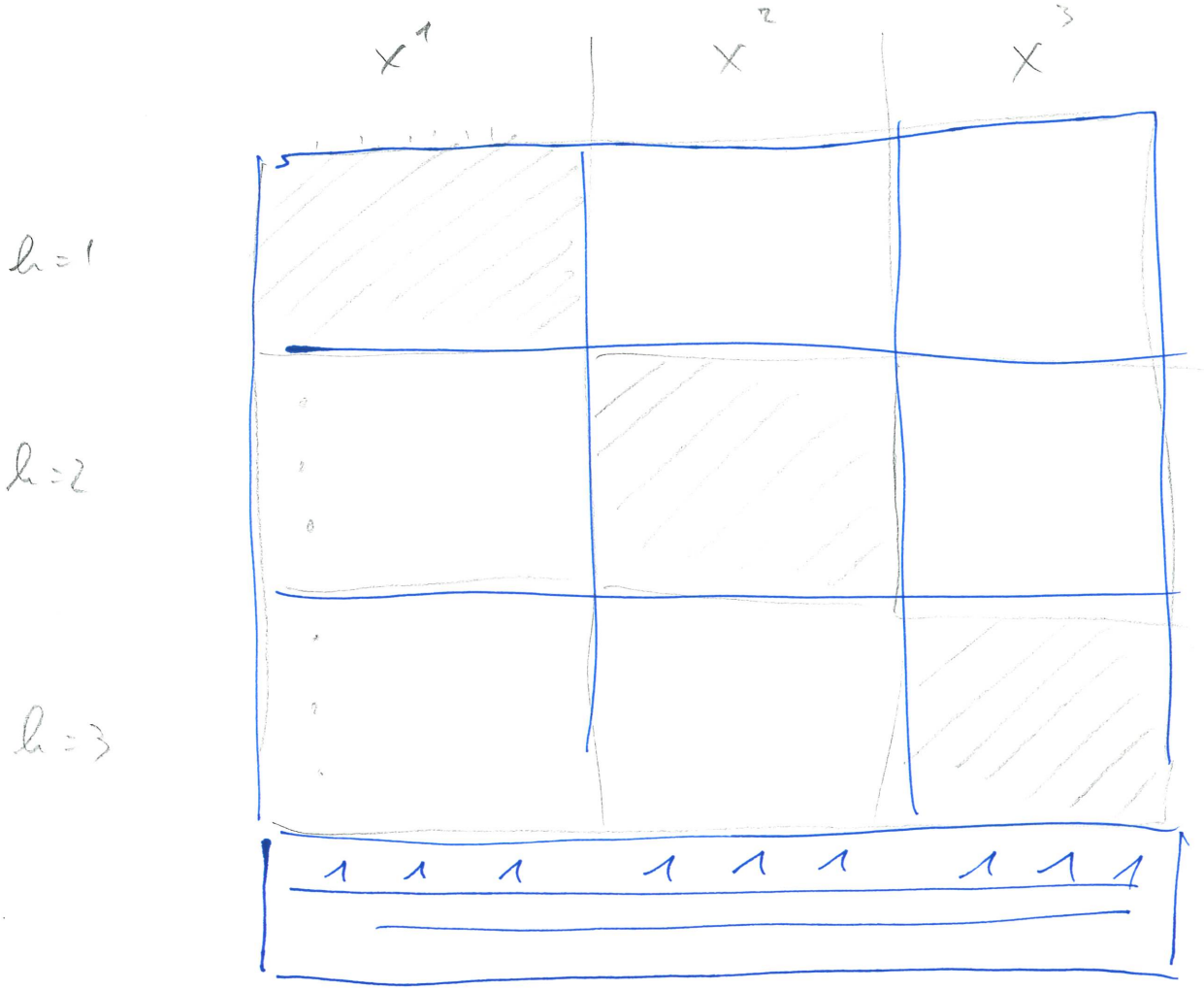


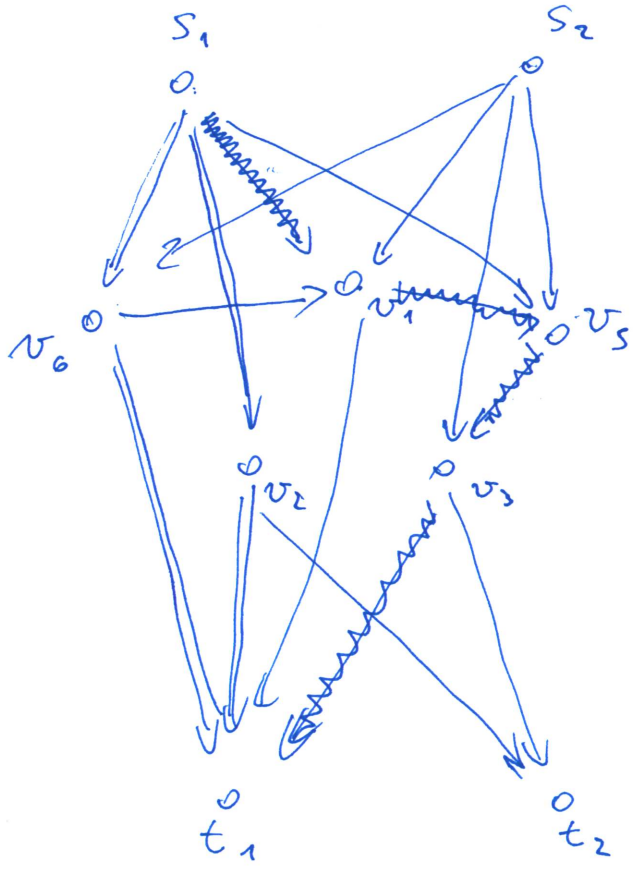
$$z_{LR} = \min\left(2-\lambda_1-\lambda_2\right)x_1 + \left(3-\lambda_3\right)x_2 + \left(4-\lambda_1\right)x_3 +$$

$$+\left(4-\lambda_2-\lambda_3\right)x_4 + \lambda_1 + \lambda_2 + \lambda_3$$

$$x_i \in \{0,1\} \qquad i = 1 \dots 4$$





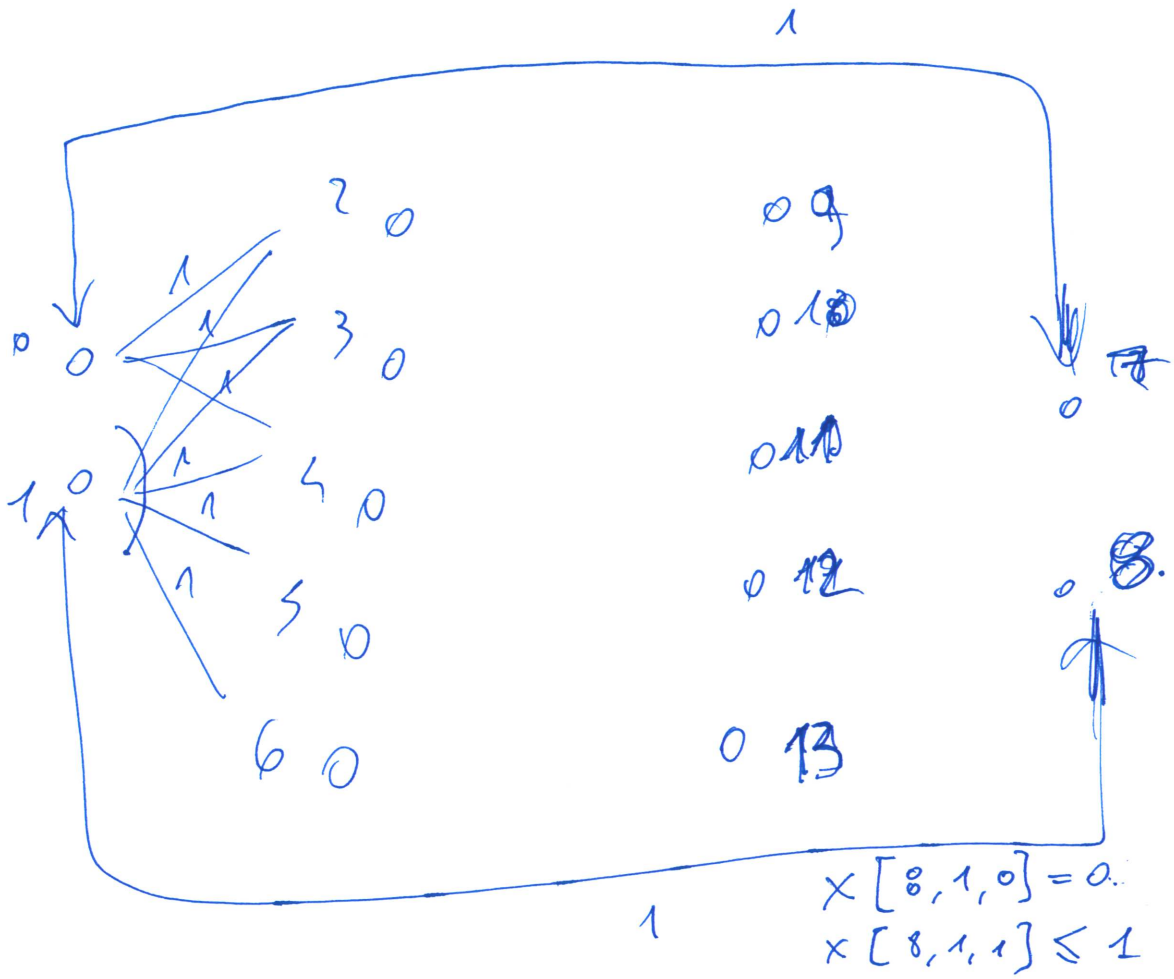


$$a_{1p} = \begin{cases} 1 & \text{if } p \text{ visits } v_1 \\ 0 & \text{otherwise} \end{cases}$$

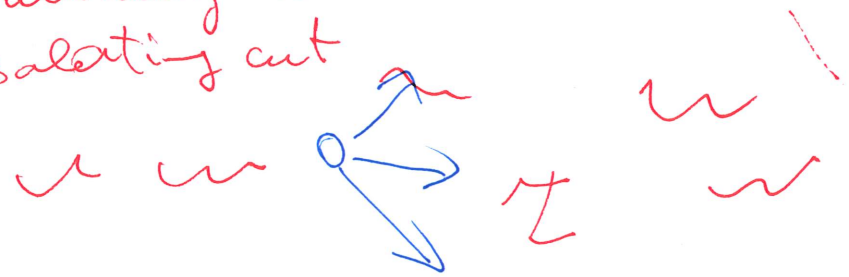
$a_{2p} =$

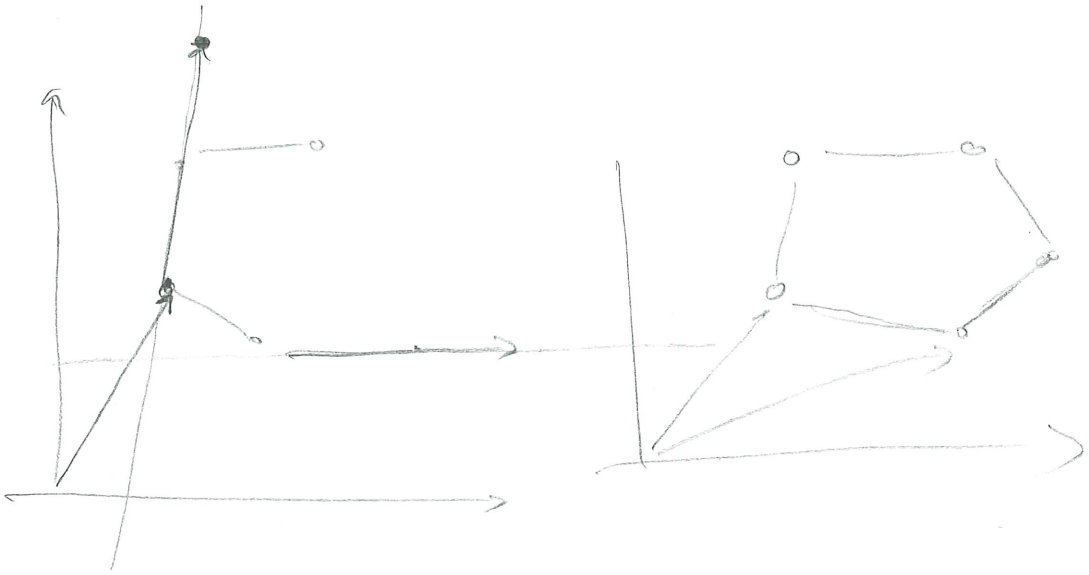
$a_{3p} =$

$$b_p^h = \begin{cases} 1 & \text{if } p \text{ starts from } h \\ 0 & \text{otherwise} \end{cases}$$



Multiway-cut
isolating cut





COMPACT F.

↓ DANTZIG-WOLF

EXTENSIVE F.
MASTER

$$\min \sum_i u_i$$
$$a_{ij} x_{ij} \leq b_i u_i$$
$$\sum x_{ij} \geq d_i$$
$$x_{ij} \in \mathbb{Z}^+$$
$$u_i \in \mathbb{B}$$

↓ LIN RELAX

LMP LINEAR MASTER

$$\min \sum_i \lambda_i \qquad \sum_i \lambda_i = 1$$
$$\sum_j a_{ij} \leq b_i \qquad \lambda_i \geq 0$$
$$\lambda_i \in \mathbb{Z}^+$$

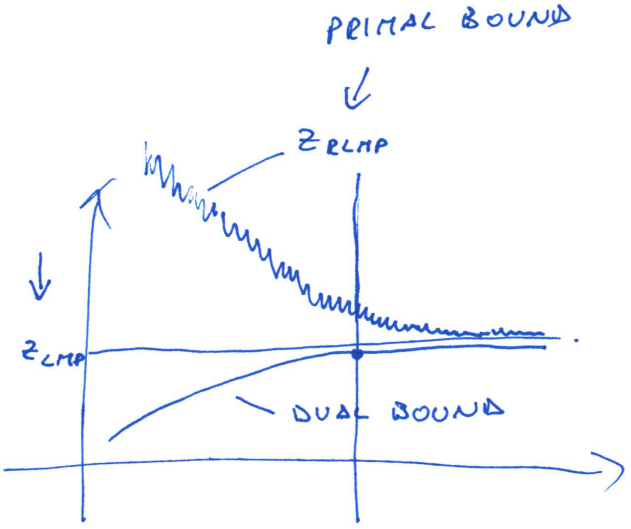
↓ Remove Same variables

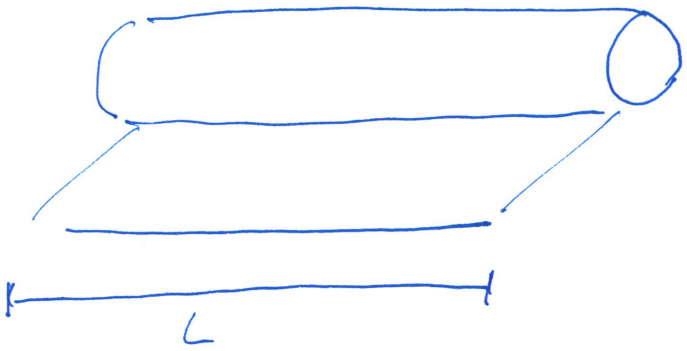
RLMP | RESTRICTED LIN. MASTER |

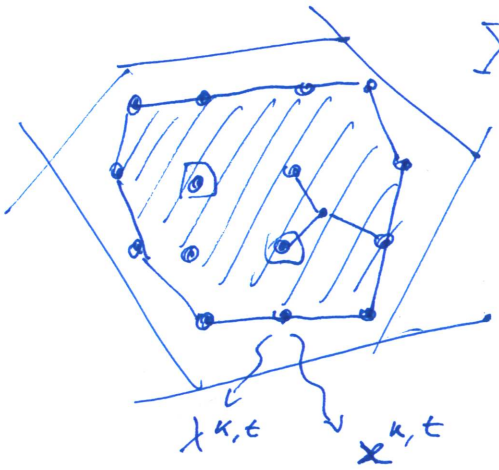
$$\min \sum \lambda_i$$
$$\sum a_{ij} \leq b_i$$
$$\lambda_i \in \mathbb{R}^+$$

↓ ↑

SUB PROBLEM
PRICING







$$X^u = \begin{cases} D^u x^u \leq d_u \\ x^u \in \mathbb{Z}_+^{m_u} \end{cases}$$

$t = 1 \dots T_u$

$x^u \in X^u$

$x^u \in \mathbb{R}^{m_u}$:

$$\begin{cases} x^u = \sum_{t \in T_u} \lambda_{u,t} \cdot x^{u,t} \\ \sum_{t=1}^{T_u} \lambda_{u,t} = 1 \\ \lambda_{u,t} \in \{0,1\}, \; t \in T_u \end{cases}$$

$$z_{MP} = \max \{ c\lambda \mid \bar{A}\lambda \leq b,\ \lambda \in Z_+^m \}$$

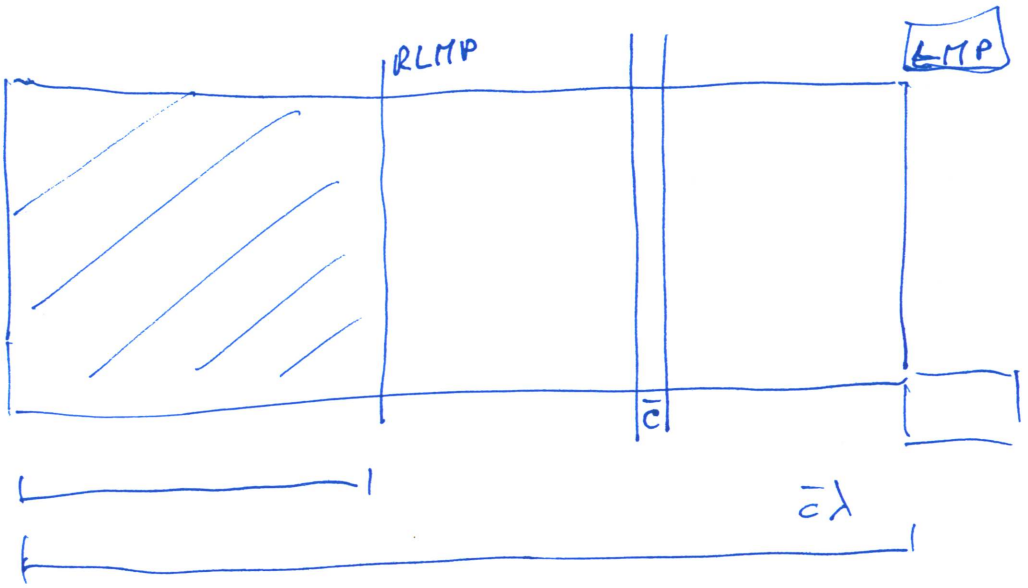
$\lambda \geq 0$

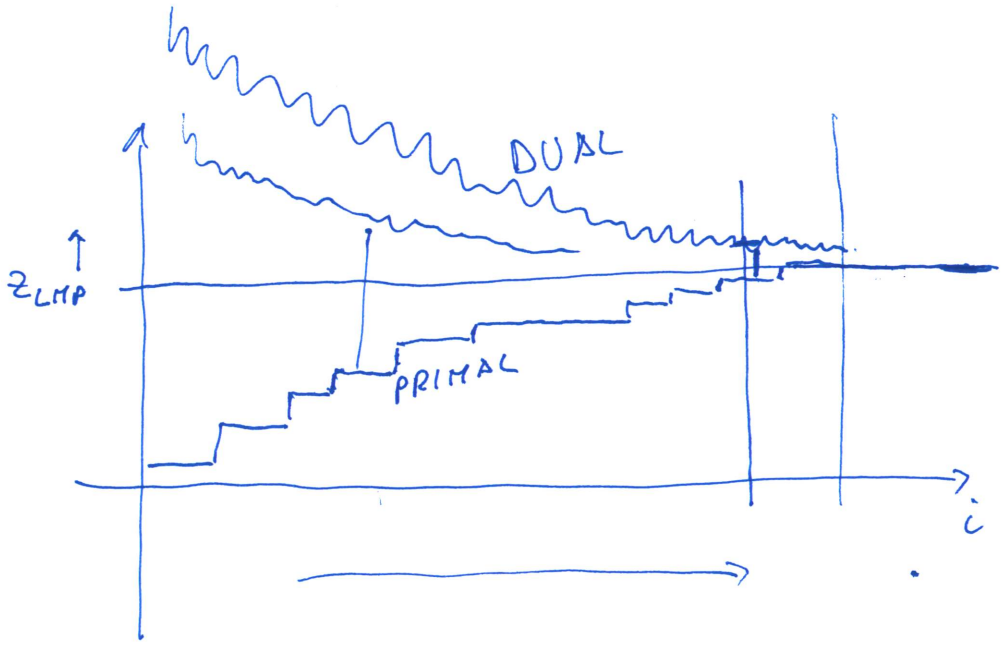
$\lambda \leq 1$

$\sum_{j=1} \lambda_j = 1$

$z_{LRMP} \leq z_{LMP}$

RLMP





$\sum_{j \in J} \lambda_j \leq K = 1$

$z_{LRMP} + z_{LMP} z_{PP} \geq z_{LMP}$

$z_{LRMP} + (1) \cdot z_{PP} \geq z_{LMP}$

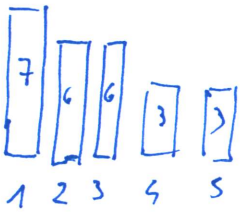
$$\min \sum_{j=1}^{U} y_j$$

$$\sum_{j=1}^{U} x_{ij} = 1 \quad \forall i \in I$$

$$\sum_{i \in I} s_i x_{ij} \leq B y_j \quad \forall j \in J$$

$$x_{ij} \in \mathbb{B} \quad \forall i \in I, j \in J$$

$$y_j \in \mathbb{B} \quad \forall j \in J$$



$T = \{1..3\}$

$d_t \in \mathbb{Z}^+, \; t \in T$

$$\min \sum_{p \in P} \Theta_p \qquad \Theta_p \in P$$

$$\sum_{p \in P} t_{ip} \, \Theta_p \geq 1 \quad \forall i \in I$$

$$\Theta_p \in \mathbb{B}^+ \quad \forall p \in P$$

$P' \subseteq P$

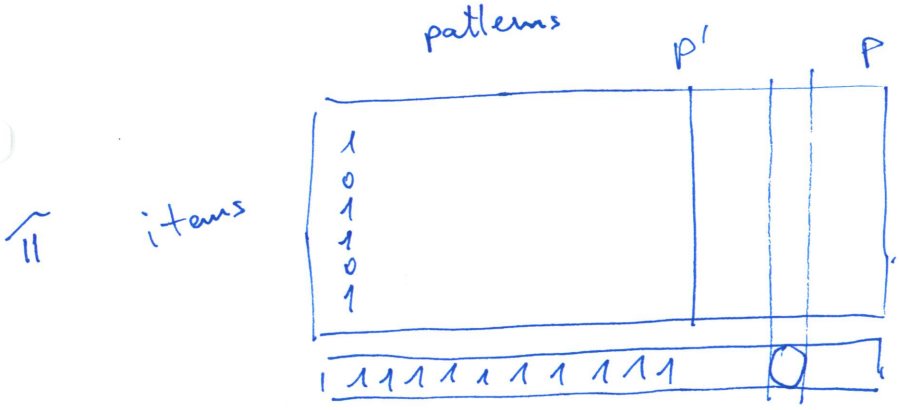
RLMP

$$\min \; 1 - \sum_{i \in I} \mu_i \alpha_i$$

$$\sum_{i=1}^{m} s_i \alpha_i \leq B$$

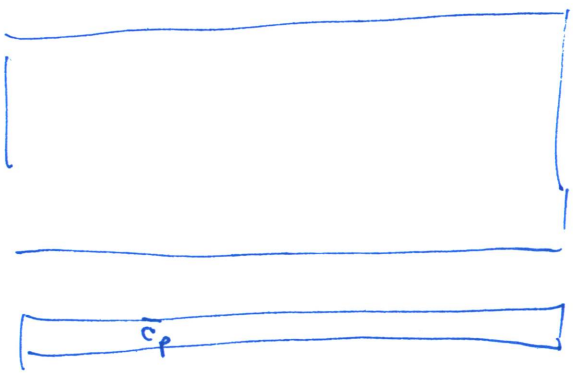
$$\alpha_i \in \mathbb{B}$$

$7\alpha_1 + 6\alpha_2 + 6\alpha_3 +$



$$z_{pp} = \max \sum_{i \in I} \mu_i \alpha_i$$

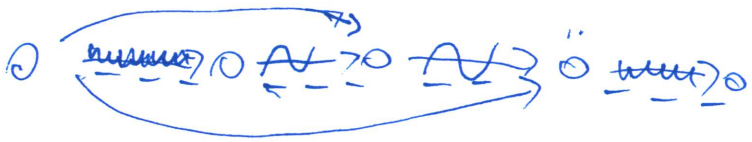
$$\sum_{i=1}^{m} \mu_i t_{ip} > 1$$



$-\mu_i$

$$\bar{c}_p = 1 + \sum_{i=1}^{m} \bar{\pi}_i t_{ip} =$$

$$= 1 - \sum_{i=1}^{m} \mu_i t_{ip} < 0$$



driver

block

$\min c^T x$

$A_1 x = b_1$

$A_2 x = b_2$

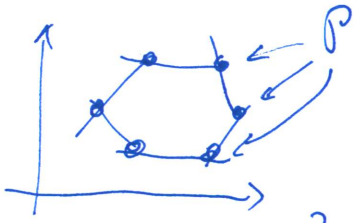
$x \geq 0$

$\min c^T x$

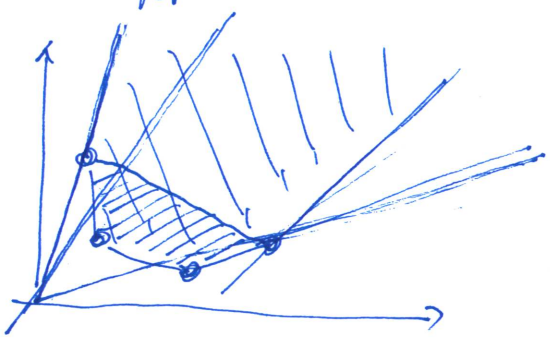
$A_1 x = b_1$

$x \in X$

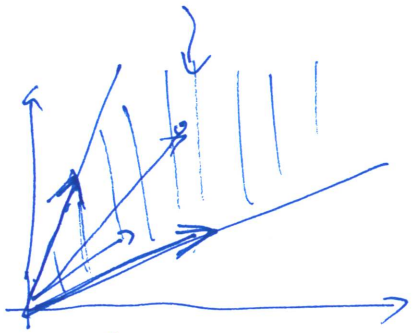
$$X = \{ x \mid A_2 x = b_2 \}$$



$$X = \left\{ x \mid x = \sum_{p \in P} \lambda_p x_p,\ \lambda_p \geq 0,\ \sum_{p \in P} \lambda_p = 1 \right\}$$



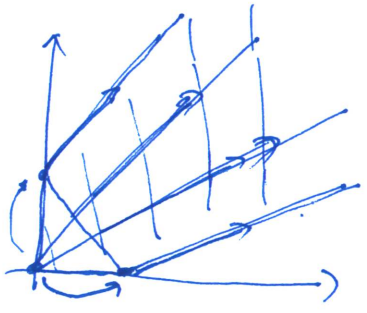
Polyhedron = Polytope + Convex Cone



$A x \leq 0$

$$X = \sum_{r \in R} \delta_r \tilde{x}_r,\ \delta_r \geq 0$$

$$X = \left\{ x \mid x = \sum_{p \in P} \lambda_p x_p + \sum_{r \in R} \delta_r \tilde{x}_r,\ \lambda_p \geq 0,\ \sum_p \lambda_p = 1,\ \delta_r \geq 0 \right\}$$



$$\min \quad c^T\left(\sum \lambda_p x_p + \sum \delta_r x_r\right)$$

$$A\left(\sum \lambda_p x_p\right) + A\left(\sum \delta_r x_r\right) = b_1$$

$$\lambda_p \geq 0$$

$$\delta_r \geq 0$$

$$\sum \lambda_p = 1$$

$$\min \quad c^T \cdot (\widetilde{11})\, x$$

$$Bx \leq b$$

$$x \geq 0$$

$$\begin{cases} x_1 + x_3 - x_4 = 5 \\ x_2 + x_3 = 5 \end{cases}$$

$x_1 = 5$ Basic

$x_2 = 5$

$x_3 = 0$ NB

~~$x_3 = 5 + x_4$~~

$x_3 =$

$x_1 = 5 + x_4$

$x_2 = 5$

$$\Rightarrow X_2 = \begin{bmatrix} 4+k \\ 5 \\ 0 \\ k \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \\ 0 \end{bmatrix} + k \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\tilde{X}_2$

(OP) $\min\ cx + f(y)$ → dy

$Ax + F(y) \geq b$ → By

$y \in Y$

$x \geq 0$

BENDERS DECOMPOSITION

(EF) $\min\ z$

$z \geq f(y) + u^p(b - F(y))$ $\quad (p = 1 \dots P)$

$u^r(b - F(y)) \leq 0$ $\quad r = 1 \dots R$

$y \in Y$

BENDERS ALGORITHM

if y is fixed to $\bar{y}$ $\Rightarrow$ $\min\ cx$ (SP)

$Ax \geq b - F(\bar{y})$

$x \geq 0$

$$\bar{y} \in \bar{Y} = \{ y \mid \exists\, x \geq 0 \text{ such that } Ax \geq b - F(\bar{y}) \}$$

if $\bar{Y} \cap Y = \phi$ $\Rightarrow$ OP is infeasible

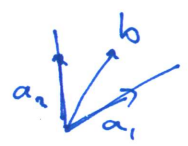
if $\bar{Y} \cap Y \neq \phi$ $\Rightarrow$ OP is solvable < bounded / unbounded

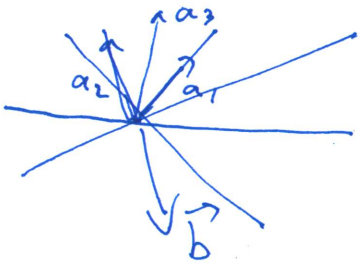
Farkas Lemma

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then:

either $\exists\, x \in \mathbb{R} : \boxed{Ax = b \text{ and } x \geq 0}$

or $\exists\, u \in \mathbb{R} : u^T A \geq 0^T$ and $u^T b > 0$

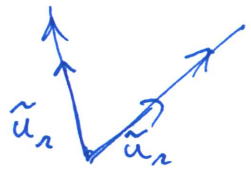




$h = \{ x \in \mathbb{R}^n : x \cdot u = 0 \}$

$\exists\, x : Ax \geq b - F(\bar{y}) \qquad x \geq 0$

$\exists\, u : \underline{uA \leq 0} \qquad u(b - F(\bar{y})) > 0 \qquad u \geq 0$



$\forall\, \tilde{u}_r : \quad \tilde{u}_r(b - F(\bar{y})) \leq 0$

$\bar{Y} = \{ y \mid \tilde{u}_r(b - F(y)) \leq 0,\ r = 1..R \}$

$$\min_y \quad f(y) + \overbrace{\min_x \{ cx \mid Ax \geq \underset{SP}{b - F(y)},\ x \geq 0 \}}^{LP}$$
$$y \in Y \cap \bar{Y}$$

$$\min_y \quad f(y) + \max_u \{ u(b - F(y)) \mid \underset{DSP}{uA \leq c},\ u \geq 0 \}$$

SP — bounded ..... DSP — bounded

SP — unbounded $= -\infty$ ..... DSP — infeasible

$$\min_y \{ f(y) + \max_{p=1..P} [u^p(b - F(y)) \mid \underline{u^p A \leq c,\ u^p \geq 0}]$$
$$y \in Y \cap \bar{Y}$$

$$\min \; z$$

$$z \geq f(y) + u^p\big(b - F(y)\big) \qquad \forall\, p = 1 \dots P$$

$$(EF) \qquad \hat{u}_r\big(b - F(y)\big) \leq 0 \qquad \forall\, r = 1 \dots R$$

$$y \in Y$$

$$\max\{3y, 4y, 5y, 6y, 7y\}$$

$$z \geq 3y$$
$$z \geq 4y$$
$$z \geq 5y$$
$$z \geq 6y$$
$$z \geq 7y$$

$\Rightarrow$ OP has a feasible sol $\Leftrightarrow$ EF has a feasible sol

$\Rightarrow$ OP is unbounded $\Leftrightarrow$ EF is unbounded

(REF) $\min z$

$z \geq f(y) + u^p (b - F(y))$ $\quad p \in P$

$\tilde{u}_r (b - F(y)) \leq 0$ $\quad r \in R$

$y \in Y$

Solve (REF) find $(\bar{z}, \bar{y})$

Solve (DSP) $\max u(b - F(\bar{y}))$

$uA \leq c$

$u \geq 0$

→ unbounded

⇓

$\exists (\tilde{u}^r)(b - F(\bar{y})) \leq 0$

→ bounded

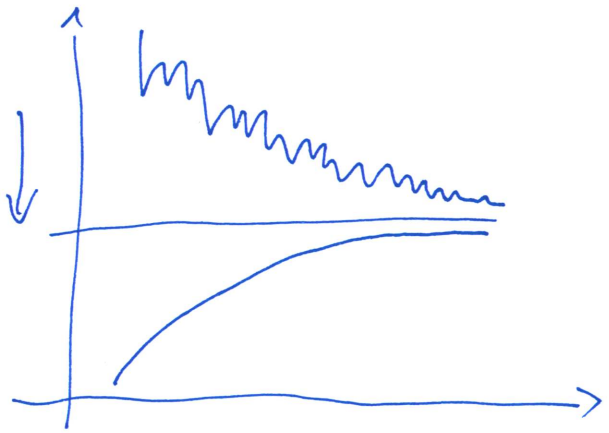
$\exists u^p$

$\bar{z} \geq f(\bar{y}) + u^p (b - F(\bar{y}))$

if $\bar{z} = f(\bar{y}) + u^p (b - F(\bar{y}))$

⇓

proof of optimality for REF



if (REF) has no feasible sol then STOP

if (REF) has unbounded value ⇒ $\bar{z} = -\infty$ and guess $\bar{y}$ and solve (DSP)

if (REF) has a bounded sol ⇒ $(\bar{z}, \bar{y})$ and solve (DSP)

if (DSP) is infeasible ⇒ STOP the prob. (OP) (EP) is unbounded

if (DSP) is unbounded ⇒ add the extreme ray constr

if (DSP) is bounded ⇒ add the extreme point constr

STOP opt found